



# Triple nontrivial radial convex solutions of systems of Monge–Ampère equations

Fanglei Wang<sup>a,b,\*</sup>, Yukun An<sup>b</sup>

<sup>a</sup> College of Science, Hohai University, Nanjing, 210098, PR China

<sup>b</sup> Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, PR China

## ARTICLE INFO

### Article history:

Received 4 March 2011

Accepted 17 July 2011

### Keywords:

System of Monge–Ampère equations

Convex radial solution

Leggett–Williams fixed point theorem

Cone

## ABSTRACT

By using the Leggett–Williams fixed point theorem, this paper investigates the existence of at least three nontrivial radial convex solutions of systems of Monge–Ampère equations.

© 2011 Elsevier Ltd. All rights reserved.

## 1. Introduction

The Monge–Ampère equations

$$\begin{cases} \det(D^2u) = \lambda f(-u), & \text{in } B \\ u(x) = 0, & \text{on } \partial B, \end{cases} \quad (1)$$

arise from the Differential Geometry and optimization and mass-transfer problems, where  $D^2u = \left( \frac{\partial u}{\partial x_i \partial x_j} \right)$  is the Hessian matrix of  $u$ ,  $B = \{x \in \mathbb{R}^N : |x| < 1\}$ . Because of its important background, there are many authors who pay more attention to the study of Eq. (1), see [1–3] and the references therein. In addition, Goncalves, Santos and Mohammed also solve the singular boundary value problems for the Monge–Ampère equations, see [4,5]. Inspired by [3], Hu, Wang recently consider the existence, multiplicity and nonexistence of convex radial solutions of Eq. (1) or Dirichlet problem for the system of the Monge–Ampère equations

$$\begin{cases} \det(D^2u_1) = \lambda f_1(-u_1, \dots, -u_n), & \text{in } B \\ \dots \\ \det(D^2u_n) = \lambda f_n(-u_1, \dots, -u_n), & \text{in } B, \\ u(x) = 0, & \text{on } \partial B, \end{cases} \quad (2)$$

where  $N \geq 1$ ,  $\lambda > 0$ ,  $D^2u_i = \left( \frac{\partial u_i}{\partial x_i \partial x_j} \right)$  is the Hessian matrix of  $u_i$ ,  $B = \{x \in \mathbb{R}^N : |x| < 1\}$ . The proof of the results is based on a fixed point theorem in a cone, see [6–9].

On the other hand, using the fixed point theorem, many authors study the multiple results of differential equations or systems, see [10–12]. Inspired by the work of the above papers, our aim in the present paper is to investigate the

\* Corresponding author at: College of Science, Hohai University, Nanjing, 210098, PR China.

E-mail addresses: [wang-fanglei@hotmail.com](mailto:wang-fanglei@hotmail.com) (F. Wang), [anykna@nuaa.edu.cn](mailto:anykna@nuaa.edu.cn) (Y. An).

existence of at least three convex radial solutions of Eq. (2) when  $\lambda = 1$ . Moreover, the method chosen in this paper is the Leggett–Williams fixed point theorem which has not been used to study Eq. (2) in the literatures.

For convenience of our station, we now introduce the following well-known Leggett–Williams fixed point theorem.

Let  $E$  be a real Banach space and  $P$  be a cone in  $E$ . A map  $\alpha$  is said to be a nonnegative continuous concave functional on  $P$  if

$$\alpha : P \rightarrow [0, +\infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all  $x, y \in P$  and  $t \in [0, 1]$ .

For numbers  $a, b$  such that  $0 < a < b$  and let  $\alpha$  be a nonnegative continuous concave functional on  $P$ , we define the following convex sets

$$P_a = \{x \in P : \|x\| < a\},$$

and

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}.$$

**Lemma 1.1** ([13] Leggett–Williams Fixed Point Theorem). Let  $T : \overline{P_c} \rightarrow \overline{P_c}$  be completely continuous and  $\alpha$  be a nonnegative continuous concave functional on  $P$  such that  $\alpha(x) \leq \|x\|$  for all  $x \in \overline{P_c}$ . Suppose there exist  $0 < d < a < b \leq c$  such that

- (i)  $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$  and  $\alpha(Tx) > a$  for  $x \in P(\alpha, a, b)$ ;
- (ii)  $\|Tx\| < d$  for  $\|x\| \leq d$ ;
- (iii)  $\alpha(Tx) > a$  for  $x \in P(\alpha, a, c)$  with  $\|Tx\| > b$ .

Then  $T$  has at least three fixed points  $x_1, x_2, x_3$  satisfying

$$\begin{aligned} \|x_1\| &< d, & a &< \alpha(x_2), \\ \|x_3\| &> d & \text{and} & \alpha(x_3) < a. \end{aligned}$$

This paper is organized as follows: In Section 2, some preliminaries are given; in Section 3, we give the main results.

## 2. Preliminaries

For radial solution  $u_i(r)$  with  $r = \sqrt{\sum_1^N x_i^2}$ , the Monge–Ampère operator simply becomes

$$\det(D^2 u_i) = \frac{(u_i')^{N-1} u_i''}{r^{N-1}} = \frac{1}{Nr^{N-1}} ((u_i')^N)',$$

and then (2) can be easily transformed into the following boundary value problem

$$\begin{cases} ((u_1'(r))^N)' = Nr^{N-1} f_1(-u_1, \dots, -u_n), & 0 < r < 1 \\ \dots \\ ((u_n'(r))^N)' = Nr^{N-1} f_n(-u_1, \dots, -u_n), & 0 < r < 1, \\ u_i'(0) = u_i(1) = 0, & i = 1, \dots, n \end{cases} \quad (MA)$$

where  $N \geq 1$ . We shall impose growth conditions on nonlinearities  $f_i$  which ensure the existence of at least three radial convex solutions for the system (MA). For convenience, with a simple transformation  $v_i(t) = -u_i(r)$ , then system (MA) can be brought to the following system

$$\begin{cases} ((-v_1'(t))^N)' = Nt^{N-1} f_1(v_1, \dots, v_n), & 0 < t < 1 \\ \dots \\ ((-v_n'(t))^N)' = Nt^{N-1} f_n(v_1, \dots, v_n), & 0 < t < 1, \\ v_i'(0) = v_i(1) = 0, & i = 1, \dots, n. \end{cases} \quad (MA')$$

Now we mainly treat the positive concave solutions of (MA').

Before stating our theorems, we make the following assumptions. Let  $R_+ = (0, +\infty)$ ,  $R_+^n = \underbrace{R_+ \times \dots \times R_+}_n$  and  $f_i$  satisfies

the following assumption

(H1)  $f_i : R_+^n \rightarrow R_+$  is continuous, and for any  $v \in R_+^n$ ,  $f_i(v) > 0$ , if  $v \neq 0$ .

Let  $E$  denote the Banach space  $\underbrace{C[0, 1] \times \cdots \times C[0, 1]}_n$ , and

$$\|v\| = \sum_{i=1}^n \sup_{t \in [0, 1]} |v_i(t)|, \quad \text{for } v(t) = (v_1(t), \dots, v_n(t)) \in E.$$

From Lemma 2.2 in [9], let  $P$  be a cone in  $E$  defined as

$$P = \left\{ v = (v_1, \dots, v_n) \in E : v_i(t) \geq 0, t \in [0, 1], i = 1, \dots, n, \text{ and } \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n v_i(t) \geq \frac{1}{4} \|v\| \right\}.$$

Let  $T : K \setminus \{0\} \rightarrow E$  be a map with components  $(T_1, \dots, T_n)$ . We define  $T_i$  by

$$T_i v(t) = \int_t^1 \varphi^{-1} \left( \int_0^s [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds, \quad 0 \leq t \leq 1, \quad (3)$$

where  $\varphi(t) = t^N$ ,  $\varphi^{-1}(t) = t^{\frac{1}{N}}$ . From a standard procedure and [9], we have

**Lemma 2.1.** Assume (H1) holds. Then  $T(K) \subset K$  and  $T : K \setminus \{0\} \rightarrow K$  is compact and continuous.

### 3. Main result

**Theorem 3.1.** Assume (H1) holds. In addition, there exist numbers  $a, c$  and  $d$  with  $0 < d < a < \frac{c}{4}$  such that the following conditions are satisfied:

(H2)  $f_i(v) < n^{-N} \varphi(d)$ , for  $t \in [0, 1]$ ,  $v_i \geq 0$  and  $0 \leq \sum_{i=1}^n v_i < d$ ;

(H3) there exists  $i_0 \in \{1, 2, \dots, n\}$ , such that

$$f_{i_0} > \frac{2^{2N}}{\Gamma^N} \varphi(a), \quad \text{for } t \in \left[ \frac{1}{4}, \frac{3}{4} \right], \quad v_i \geq 0, \quad \text{and} \quad \sum_{i=1}^n v_i \in [a, 4a],$$

$$\text{where } \Gamma = \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \int_{\frac{1}{4}}^s N \tau^{N-1} d\tau \right)^{\frac{1}{N}} ds > 0;$$

(H4)  $f_i(v) \leq n^{-N} \varphi(c)$ , for  $t \in [0, 1]$ ,  $v_i \geq 0$  and  $0 \leq \sum_{i=1}^n v_i < c$ .

Then the boundary value problem (MA') has at least three radial concave solutions.

**Proof.** For  $v = (v_1, \dots, v_n) \in P$ , define

$$\alpha(v) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n v_i(t),$$

then it is easy to know that  $\alpha$  is a nonnegative continuous concave functional on  $P$  with  $\alpha(v) \leq \|v\|$  for  $v \in P$ .

Set  $b = 4a$ . First, we show that  $T : \overline{P_c} \rightarrow \overline{P_c}$ , where  $c > b$ .

In fact, for any  $v \in \overline{P_c}$ , then  $\|v\| \leq c$  and by (3), (H4), we have

$$\begin{aligned} \|T_i v(t)\| &= \sup_{t \in [0, 1]} \int_t^1 \varphi^{-1} \left( \int_0^s [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\ &\leq \int_0^1 \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\ &\leq \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) \\ &\leq c \varphi^{-1}(n^{-N}) \\ &\leq \frac{c}{n}. \end{aligned}$$

So  $\|Tv\| = \sum_{i=1}^n \|T_i v(t)\| \leq c$ .

In the similar way, we also can prove that  $T : \overline{P_d} \rightarrow P_d$ . Then (ii) of Lemma 1.1 holds.

Next, we shall show that (i) of Lemma 1.1 is satisfied. It is clear to see that  $v = (\frac{a+b}{4}, \dots, \frac{a+b}{4}) \in \{v = (v_1, \dots, v_n) \in P(\alpha, a, b) : \alpha(v) > a\}$ . Then for any  $v \in P(\alpha, a, b)$  and  $t \in [\frac{1}{4}, \frac{3}{4}]$ , it is easy to obtain that

$$b \geq \sum_{i=1}^n \sup_{t \in [0, 1]} |v_i(t)| \geq \sum_{i=1}^n v_i(t) \geq \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n v_i(t) = \alpha(v) > a.$$

Then by (H3), we can have

$$\begin{aligned}
 \alpha(Tv(t)) &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n T_i v(t) \\
 &\geq \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} T_{\lambda_0} v(t) \\
 &\geq \frac{1}{4} \|T_{\lambda_0} v(t)\| \\
 &= \frac{1}{4} \sup_{0 \leq t \leq 1} |T_{\lambda_0} v(t)| \\
 &\geq \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1} \left( \int_{\frac{1}{4}}^s [N \tau^{N-1} f_{i_0}(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\
 &> \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1} \left( \int_{\frac{1}{4}}^s \left[ N \tau^{N-1} \frac{2^{2N}}{F^N} \varphi(a) \right] d\tau \right) ds \\
 &= a.
 \end{aligned}$$

Finally, we will verify that (iii) of Lemma 1.1 is satisfied. Suppose that  $v \in P(\alpha, a, c)$  with  $\|Tv\| > b$ , then we can have

$$\alpha(Tv(t)) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n T_i v(t) \geq \sum_{i=1}^n \frac{1}{4} \|T_i v\| \geq \frac{1}{4} \|Tv\| > \frac{b}{4} = a. \quad (4)$$

Above all, the hypotheses of the Leggett–Williams theorem are satisfied. Hence (MA') has at least three positive solutions  $v^1 = (v_1^1, \dots, v_n^1)$ ,  $v^2 = (v_1^2, \dots, v_n^2)$  and  $v^3 = (v_1^3, \dots, v_n^3)$  such that  $\|v^1\| < d$ ,  $a < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n v_i^2(t)$ , and  $\|v^3\| > d$  with

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^n v_i^3(t) < a.$$

The proof is complete.  $\square$

**Theorem 3.2.** Assume that (H1) and (H3) hold. In addition, suppose that the following conditions are satisfied

(H5)

$$\lim_{\|v\| \rightarrow 0^+} \sup_{t \in [0, 1]} \frac{f_i(v)}{\varphi(\|v\|)} < n^{-N};$$

(H6)

$$\lim_{\|v\| \rightarrow +\infty} \sup_{t \in [0, 1]} \frac{f_i(v)}{\varphi(\|v\|)} \leq \frac{1}{2} n^{-N}.$$

Then the boundary value problem (MA') has at least three radial concave solutions.

**Proof.** First, we shall show that there exists a positive number  $c$  with  $c \geq b = 4a$  such that  $T : \overline{P_c} \rightarrow \overline{P_c}$ .

By (H6), there exists  $\delta > 0$  such that

$$f_i(v) \leq \frac{1}{2} n^{-N} \varphi(\|v\|), \quad \forall t \in [0, 1], \|v\| \geq \delta.$$

Set

$$M_i = \max\{f_i(v) : t \in [0, 1], v \in [0, \delta]\}.$$

Taking

$$c > \max \left\{ b, n(2M_1)^{\frac{1}{N}}, \dots, n(2M_n)^{\frac{1}{N}} \right\}.$$

If  $v \in \overline{P_c}$ , then we have

$$\|Tv\| = \sum_{i=1}^n \|T_i v(t)\|$$

$$\begin{aligned}
&= \sum_{i=1}^n \sup_{t \in [0,1]} \int_t^1 \varphi^{-1} \left( \int_0^s [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\
&\leq \sum_{i=1}^n \int_0^1 \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\
&\leq \sum_{i=1}^n \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) \\
&\leq \sum_{i=1}^n \varphi^{-1} \left( \int_0^1 \left[ N \tau^{N-1} \left( \frac{1}{2} n^{-N} \varphi(\|v\|) + M_i \right) \right] d\tau \right) \\
&\leq \sum_{i=1}^n \frac{c}{n} = c.
\end{aligned}$$

So  $\|Tv\| = \sum_{i=1}^n \|T_i v(t)\| \leq c$ .

Next, from (H5), there exists  $d \in (0, a)$  such that

$$f_i(v) < n^{-N} \varphi(\|v\|), \quad \forall t \in [0, 1], \quad \|v\| \in [0, d].$$

For each  $u \in \overline{P_d}$ , we also have

$$\begin{aligned}
\|Tv\| &= \sum_{i=1}^n \|T_i v(t)\| \\
&= \sum_{i=1}^n \sup_{t \in [0,1]} \int_t^1 \varphi^{-1} \left( \int_0^s [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\
&\leq \sum_{i=1}^n \int_0^1 \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) ds \\
&\leq \sum_{i=1}^n \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} f_i(v_1(\tau), \dots, v_n(\tau))] d\tau \right) \\
&< \sum_{i=1}^n \varphi^{-1} \left( \int_0^1 [N \tau^{N-1} (n^{-N} \varphi(\|v\|))] d\tau \right) \\
&\leq \sum_{i=1}^n \frac{d}{n} = d.
\end{aligned}$$

Finally, the left proof is similar to [Theorem 3.1](#), we omit it.  $\square$

## References

- [1] L. Caffarelli, Y. Li, An extension to a theorem of Jrgens, Calabi, and Pogorelov, *Comm. Pure Appl. Math.* 56 (2003) 549–583.
- [2] C. Gutierrez, *The Monge–Ampère Equation*, Birkhuser, Basel, 2000.
- [3] Z. Zhang, K. Wang, Existence and non-existence of solutions for a class of Monge–Ampère equations, *J. Differential Equations* 246 (2009) 2849–2875.
- [4] J.V.A. Goncalves, C.A.P. Santos, Classical solutions of singular Monge–Ampère equation in a ball, *J. Math. Anal. Appl.* 305 (2005) 240–252.
- [5] A. Mohammed, Singular boundary value problems for the Monge–Ampère equation, *Nonlinear Anal.* 70 (2009) 457–464.
- [6] S. Hu, H. Wang, Convex solutions of boundary value problems arising from Monge–Ampère equations, *Discrete Contin. Dyn. Syst.* 16 (2006) 705–720.
- [7] H. Wang, Convex solutions of boundary value problems, *J. Math. Anal. Appl.* 318 (2006) 246–252.
- [8] H. Wang, Convex solutions of systems arising from Monge–Ampère equations, *Electron. J. Qual. Theory Differ. Equ.* 26 (2009) 1–8 (special edition I).
- [9] H. Wang, Convex solutions of systems of Monge–Ampère equations, [arXiv:1007.3013v2](#).
- [10] J.M. Davis, P.W. Eloe, J. Henderson, Triple positive solutions and dependence on high order derivatives, *J. Math. Anal. Appl.* 237 (1999) 710–720.
- [11] F. Wang, Y. An, Triple positive doubly periodic solutions of a nonlinear telegraph system, *Appl. Math. Mech. (English Ed.)* 30 (1) (2009) 81–88.
- [12] Q. Yao, Positive solutions to a semilinear system of second order two-point boundary value problems, *Ann. Differential Equations* 22 (1) (2006) 87–94.
- [13] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.